k is the Kepler number, d_w the drag-to-weight ratio, and p a propulsion parameter; each of these is positive. It is clear that

$$T_s - D \neq 0 \tag{17}$$

under any circumstance. It is noteworthy that $\gamma = 0$ (a necessary condition for a forced circular orbit) does not lie on the singular surface since, as $\gamma \to 0$, $T_s \to \infty$. This is a result of the preimposed steering law $\mathbf{T}/T = \mathbf{v}/v$. In fact, this steering law alters the structure of the singular arc. This is because since the order of the singular arc (half the number of repeated differentiations of the switching function) for the exoatmospheric case is two,[6] we must have $[a_1, [a_0, a_1]] = 0$ [see Eq. (11)] whenever $D = 0$. This is equivalent to the condition[3]

$$T_s \to \infty \qquad \text{as} \qquad D \to 0 \tag{18}$$

However, from Eqs. (16), it is clear that

$$T_s \to mgs\gamma \qquad \text{as} \qquad D \to 0 \tag{19}$$

is a finite quantity. Thus, the steering law has created an "artificial" singular arc.

Fuel Efficiency of Orbit Raising

From the previous sections, an FKT is not an extremal arc of the Mayer-optimal control problem, and hence it cannot be a subarc of an optimal trajectory. Accordingly, for orbit maintenance, periodic boosting must provide fuel-efficient trajectories. Although this result indicates what is not optimal, it does not however tell us how the reboost maneuvers must be performed or whether the thrusting is singular or maximum. For the purpose of exploring the utility of this result, we wish to compare analytically the fuel required for an FKT with a periodic Hohmann transfer by way of a linear analysis.

Suppose the orbit of a spacecraft contracts (due to atmospheric drag) from its initial circular orbit of radius r down to $r - \Delta r$ in time $t_D \gg \tau = 2\pi r/v$, where τ is a first-order approximation to the orbital period. To perform a Hohmann transfer from $r_1 = r - \Delta r$ to $r_2 = r$, the required Δv is given by[1]

$$\Delta v = \sqrt{\frac{\mu}{r_1}} \left[\sqrt{\frac{2(r_2/r_1)}{1 + r_2/r_1}} \left(1 - \frac{r_1}{r_2}\right) + \sqrt{\frac{r_1}{r_2}} - 1 \right] \cong v\frac{\Delta r}{2r} \tag{20}$$

where we have assumed that $\Delta r/r \ll 1$ and $v = \sqrt{\mu/r_1}$. Thus, the Δv budget per unit time for a periodic Hohmann transfer can be approximated by

$$\Delta v_{\text{Hohmann}} \cong \Delta v/t_D = v\,\Delta r/2rt_D \tag{21}$$

The first-order change in orbital radius per orbit is given by[1]

$$\delta r = 2\pi C_D A \rho r^2/m \tag{22}$$

Since $\Delta r = \delta r\, t_D/\tau$, Eq. (21) reduces to

$$\Delta v_{\text{Hohmann}} = v\,\delta r/2r\tau = D/m \tag{23}$$

The first-order change in orbital speed per orbit is given by[1]

$$\delta v = \pi C_D A \rho r v/m \tag{24}$$

Thus, the Δv budget per unit time for an FKT can be approximated by

$$\Delta v_{\text{FKT}} \cong \delta v/\tau = D/m = \Delta v_{\text{Hohmann}} \tag{25}$$

Conclusions

An FKT is not a fuel-optimal maneuver since it is not a singular arc. For LEO maintenance, any savings in propellant accomplished by a periodic Hohmann maneuver is a higher order effect. Since the optimal maneuver is unknown, we have at least two possibilities: (1) the propellant consumed by an FKT is close to the optimal if a periodic Hohmann maneuver is also close to the optimal or (2) since the propellant consumed by a Hohmann-type maneuver is close to the nonoptimal FKT, the periodic optimal maneuver is quite different from the Hohmann maneuver, possibly consisting of singular subarcs. One way to resolve this question is to develop a minimum-fuel, finite-burn maneuver by using periodic optimal control theory and compare its performance to that of an FKT.

Acknowledgment

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Near-Optimal Three-Dimensional Rotational Maneuvers of Spacecraft Using Manipulator Arms

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Introduction

THIS Note presents a framework for obtaining near-optimal three-dimensional rotational maneuvers of spacecraft possessing multiple interconnected manipulator arms. In the absence of external torques, the angular momentum of the spacecraft is conserved and this imposes a constraint on its motion. The nonholonomic nature of this constraint allows for the design of open-loop control profiles for positioning the spacecraft attitude as well as the joint angles, as shown by Reyhanoglu and McClamroch[1] on a planar example. The method was extended to three-dimensional maneuvers by Mukherjee and Zurowski.[2]

Two types of maneuvers are presented in this Note. The first one minimizes joint accelerations with specified final time and the second one minimizes the maneuver time with specified bounds on the

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joint accelerations. For both types of maneuvers, the control variables are parameterized and a sequential quadratic programming (SQP) algorithm is used to optimize the parameters. The maneuvers obtained are near optimal since they closely approximate the actual optimal maneuvers that would have been obtained from classical variational principles. Since the near-optimal maneuvers discussed here are represented by just a few parameters, they are numerically easier to obtain than the actual optimal maneuvers. The problem of obtaining three-dimensional maneuvers for space-based manipulator systems has been previously addressed by Fernandez et. al.[3] and by Hussain and Kane.[4] These researchers looked at optimizing the joint trajectories for specified spacecraft attitude maneuvers in an energy sense. This Note makes a comparison in the system performance between minimum-time and minimum-joint-acceleration maneuvers. The motivation for considering joint accelerations instead of joint torques is also given.

System Dynamics

Consider a spacecraft with two interconnected links as shown in Fig. 1. Each of the links has two degrees of freedom, ϕ and ψ. The method for obtaining the equations of motion using the conservation of angular momentum constraint is demonstrated as follows: Denote the vector $x \equiv [\phi_1 \quad \psi_1 \quad \phi_2 \quad \psi_2]^T$. The orientation of the links is described by two unit vectors R_1 and R_2:

$$\begin{aligned} R_1 &= [C\psi_1 C\phi_1 \quad S\psi_1 C\phi_1 \quad S\phi_1]^T \\ R_2 &= [C\psi_2 C\phi_2 \quad S\psi_2 C\phi_2 \quad S\phi_2]^T \end{aligned} \tag{1}$$

where $C\bullet$ and $S\bullet$ denote $\cos(\bullet)$ and $\sin(\bullet)$, respectively. Let m_0, m_1, and m_2 be the masses of the spacecraft and the links, respectively. Since there are no external forces on the system, the center of mass of the overall system is inertially fixed. The individual center-of-mass locations of the spacecraft as well as the links with respect to this inertially fixed overall center of mass are denoted by r_0, r_1, and r_2. These position vectors can be easily expressed in terms of the orientation vectors R_1 and R_2 and the masses m_0, m_1, and m_2. The velocities of the individual center-of-mass locations are given as

$$\begin{aligned} v_0 &= \left[\frac{\mathrm{d}r_0}{\mathrm{d}x}\right]\dot{x} + \omega \times r_0, \qquad v_1 = \left[\frac{\mathrm{d}r_1}{\mathrm{d}x}\right]\dot{x} + \omega \times r_1 \\ v_2 &= \left[\frac{\mathrm{d}r_2}{\mathrm{d}x}\right]\dot{x} + \omega \times r_2 \end{aligned} \tag{2}$$

where ω is the spacecraft angular velocity vector. The complete angular momentum of the system is now given as

$$h = I_0\omega + m_0 r_0 \times v_0 + h_1 + h_2 \tag{3}$$

where I_0 is the moment of inertia of the spacecraft and h_1 and h_2 are the angular momenta of the links expressed in the spacecraft's coordinate frame. In calculating h_1 and h_2, it is more convenient to express them in the link-fixed coordinate frames and then transform them to the spacecraft coordinate frame. The angular momenta of the links expressed in their respective coordinate frames are

$$h_1' = I_1\omega_1 + m_1 r_1 \times v_1, \qquad h_2' = I_2\omega_2 + m_2 r_2 \times v_2 \tag{4}$$

where ω_1 and ω_2 are the angular velocities of the links in their respective link-fixed coordinate frames. These angular velocities can be expressed in terms of the spacecraft angular velocity ω, the joint angles, and the joint rates. The link-fixed angular momentum vectors calculated in Eq. (4) can be transformed into the spacecraft-fixed frame using direction cosine matrices involving the joint angles. If it is assumed that there are no external torques present and that the space-based manipulator system is initially not moving, then the total angular momentum given by Eq. (3) will be identically zero. This angular momentum can be expressed as

$$J(\phi, \psi)\omega + H(\phi, \psi)\begin{bmatrix}\dot{\phi} \\ \dot{\psi}\end{bmatrix} = 0 \tag{5}$$

The above expression relates the spacecraft angular velocity vector ω to the joint velocities. Note that the above equations do not involve the joint torques. In order to incorporate the joint torques into the system dynamics, the complete Lagrange equations of motion will have to be formulated, and these equations are much more complicated than the conservation of angular momentum equations shown above. It is for this reason that the optimal maneuvers presented below use joint accelerations as the control variables instead of joint torques.

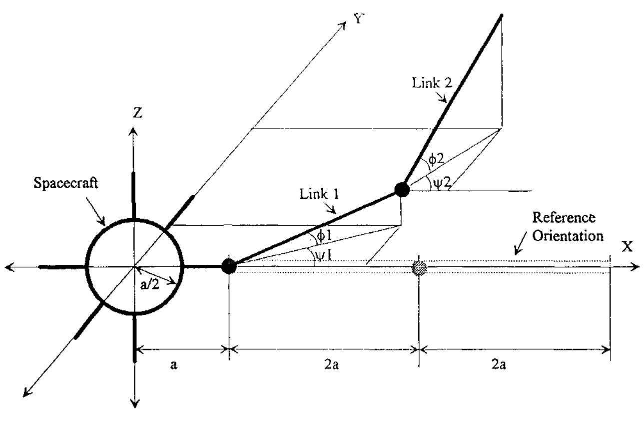


Fig. 1 Schematic of three-dimensional spacecraft with manipulator arms.

Minimum-Joint-Acceleration Maneuver

For this maneuver, the joint accelerations are parameterized as polynomials in time, and the performance index is the L_2 norm of the acceleration variables as shown in the following expression:

$$J = \frac{1}{2}\int_0^{tf} \left(\ddot{\psi}_1^2 + \ddot{\phi}_1^2 + \ddot{\psi}_2^2 + \ddot{\phi}_2^2\right) \mathrm{d}t \tag{6}$$

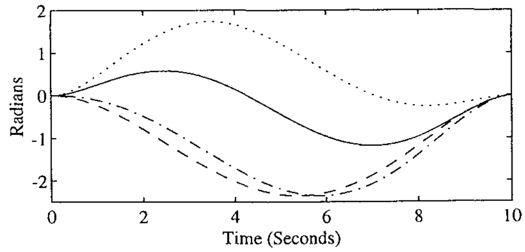
The spacecraft is initially at the reference configuration shown in Fig. 1. To describe the spacecraft attitude, a 3–2–1 (yaw–pitch–roll) Euler angle set was employed. It is desired that the spacecraft undergo a pure roll rotation of 0.5 rad at the end of the maneuver. All the joint angles are required to be zero at the initial and final times. It can be seen from the schematic that a pure roll maneuver is indeed a three-dimensional maneuver requiring changes in all the joint angles in the system. In contrast, pure yaw and pitch rotations are planar maneuvers requiring changes in only the ϕ variables for the former case and only the ψ variables for the latter case. The maneuver time is selected to be 10 s. Also, the links must not be allowed to "hit" the spacecraft or "fold" onto themselves. These conditions can be conservatively written as

$$|\phi_1|, \ |\psi_1| \le 3\pi/4 \tag{7a}$$

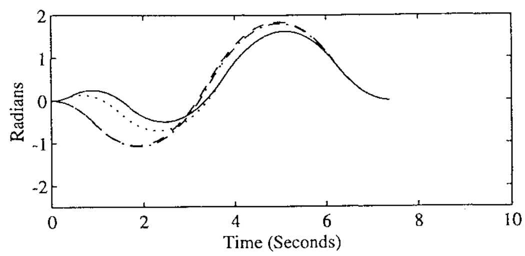
$$|[1\,0\,0]^T + 2R_1 + 2R_2| \ge 1 \tag{7b}$$

$$R_1^T R_2 \ge -0.8 \tag{7c}$$

With these specifications, the parameters for the polynomial representations of the joint angles are optimized using the SQP routine NCONF available in the International Mathematics and Statistics Library (IMSL). A total of 16 parameters were used, 4 for each joint acceleration. The results for the minimum-acceleration maneuver are given in Figs. 2a, 3a, and 4a, which show the joint angle, spacecraft attitude, and joint acceleration profiles, respectively. It can be seen that these profiles meet the specifications given above. It must be remembered that the maneuver obtained as a result of this optimization is only near optimal since a finite number of parameters was used. However, as the number of parameters is increased, the corresponding solution will approach the actual optimal solution. For this particular problem, it was observed that any further increase in the number of parameters does not significantly reduce the performance index. Hence it was concluded that the 16-parameter solution presented above is a close approximation of the actual optimal solution.

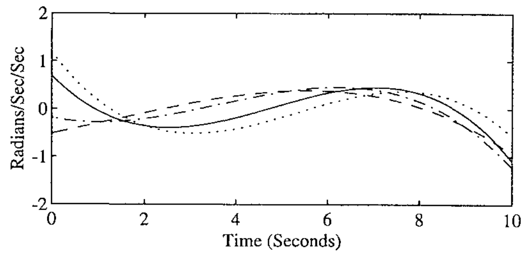


a) Minimum joint acceleration maneuver

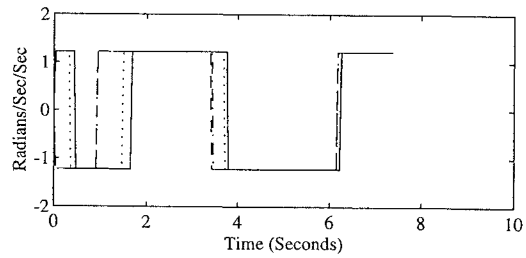


b) Minimum time maneuver

Fig. 2 Joint angle profiles; - - - psi1, — phi1, · · · psi2, -.-.- phi2.

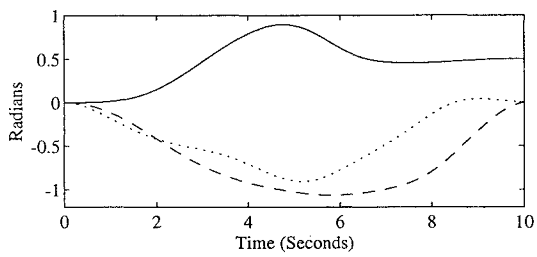


a) Minimum joint acceleration maneuver

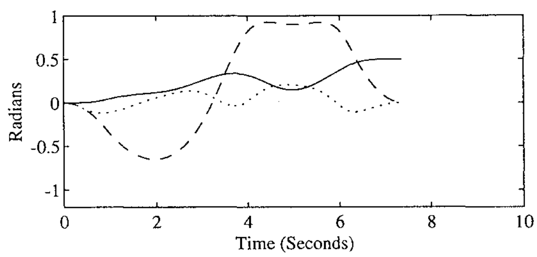


b) Minimum time maneuver

Fig. 4 Joint acceleration profiles; - - - psi1, — phi1, · · · psi2, -.-.- phi2.



a) Minimum joint acceleration maneuver



b) Minimum time maneuver

Fig. 3 Spacecraft attitude profiles; - - - roll, — pitch, · · · yaw.

Minimum-Time Maneuver

In order to appropriately pose the minimum-time control problem, a bound on the control variables (joint accelerations) is required. To make a comparison between the minimum-joint-acceleration and minimum-time maneuvers, it is stipulated that the maximum joint accelerations attained for both types of maneuvers be the same. It can be seen from Fig. 4a that the maximum joint acceleration attained for the minimum-acceleration maneuver is about 1.22 rad/s/s. This value is used as the bound for the minimum-time maneuver. From Pontryagin's principle, it can be shown that, in the absence of singular intervals, the minimum-time controls will always saturate at this bound. Hence the controls can be parameterized by a constant magnitude and a set of switch times:

$$u_i = u_{i0}\,\mathrm{sign}[\Pi(t - s_{ij})] \qquad i = 1, 4, \qquad j = 1, \ldots, n \tag{8}$$

where u_i denotes the joint accelerations, u_{i0} represents the constant magnitudes, and s_{ij} is the set of switch times. The magnitudes and switch times were optimized using SQP with the same specifications as the minimum-joint-acceleration maneuver. The total number of switches for the maneuver was set at 12 with every joint acceleration variable having three switches each. The results for the minimum-time maneuver are given in Figs. 2b, 3b, and 4b, which show the joint angle, spacecraft attitude, and joint acceleration profiles, respectively. Note that all the joint accelerations saturate at the bound provided (1.22 rad/s/s), which indicates that the 12-switch maneuver does indeed represent the actual minimum-time solution. It can be shown from optimal control principles that the minimum-time solution requires that the control variables saturate at their specified bounds. One can recognize a non-minimum-time solution as one where the control variables do not saturate at their specified bounds at any time during the maneuver. This occurs when the total number of switches for the maneuver is not sufficient. For this particular problem, it was observed that 12 switches is the minimum necessary for ensuring that the control variables saturate at the specified bounds. It can be seen from Figs. 2b and 3b that the specifications for the pure roll maneuver are satisfied. The maneuver time is approximately 7.4 s, about 2.6 s less than the minimum-joint-acceleration maneuver.

Conclusion

The Note presents two near-optimal maneuvers of spacecraft possessing multiple interconnected manipulator arms. The first maneuver minimizes joint accelerations and the second one minimizes the maneuver time. For both these maneuvers, the control variables were the joint accelerations instead of the joint torques. This enabled the use of simpler conservation of angular momentum equations in the optimization rather than having to consider the more complicated Lagrange equations of motion.

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